DISCRETE TIME EPIDEMICS MODELS

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Epidemics In Strongly Fluctuating Populations

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Demographic Equation

T(n): population size at generation n

f: models nonlinear birth or recruitment process

γ: constant probability of surviving

$$T(n+1)=f(T(n))+\gamma\, T(n) \qquad (1)$$

Examples of Demography

1) Constant rate Λ: $f(T(n))=\Lambda$

2) Constant per-capita recruitment rate μ: $f(T(n))= \mu T(n)$

3) Density dependent per-capita growth rate: $f(T(n))=T(n)g(T(n))$, where the per-capita growth function $g: [0, \infty) \rightarrow (0, \infty)$ is a strictly decreasing positive, smooth function with $\lim_{T\rightarrow\infty} g(T)<1$.

Demographic Equation

T(n): population size at generation n

f: models nonlinear birth or recruitment process

γ: constant probability of surviving

$$T(n+1)=f(T(n))+\gamma T(n) \qquad (1)$$

Asymptotically Bounded Growth

Equation (1) with constant rate Λ and initial condition T_0 gives rise to the following

$T(n+1)= \gamma T(n)+\Lambda,\ \ T(0)=T_0$

Since

$T(1)=\gamma\, T_0 +\Lambda,$

$T(2)=\gamma^2\, T_0 +(\gamma+1)\, \Lambda,$

$T(3)=\gamma^3\, T_0 +(\gamma^2+\gamma+1)\, \Lambda,\ \ \ldots,$

$T(n)=\gamma^n\, T_0 +(\gamma^{n-1}+\gamma^{n-2}+\ldots+\gamma+1)\, \Lambda$

Asymptotically Bounded Growth

The solution to Model (1) is given by the population sequence $\{T(n)\}_{n \geq 0}$

where

$$T(n)=\begin{cases} T_0+\Lambda n & \text{if } \gamma=1, \\ \gamma^n\left(T_0-\dfrac{\Lambda}{1-\gamma}\right)+\dfrac{\Lambda}{1-\gamma} & \text{if } \gamma\neq 1 \end{cases}$$

$0<\gamma<1$ imply that

- $T(n) \rightarrow \Lambda/(1-\gamma)$ as $n \rightarrow \infty$.
- $T_\infty = \Lambda/(1-\gamma)$

Geometric Growth

If new recruits arrive at the positive per-capita rate μ per generation, that is, if $f(T(n))=\mu T(n)$ then
$T(n+1)=(\mu+\gamma)T(n)$,
that is,
$T(n)=(\mu+\gamma)^n T(0)$.

The demographic basic reproductive number is $R_d=\mu/(1-\gamma)$
R_d, a dimensionless quantity, gives the average number of descendants produced by a small pioneer population (T(0)) over its life-time.

$R_d>1$ implies that the population invades at a geometric rate.
$R_d<1$ leads to extinction.

Density Dependent Per Capita Growth Rate

If

$f(T(n))=T(n)g(T(n))$,

then

$T(n+1)=T(n)g(T(n))+\gamma T(n)$.

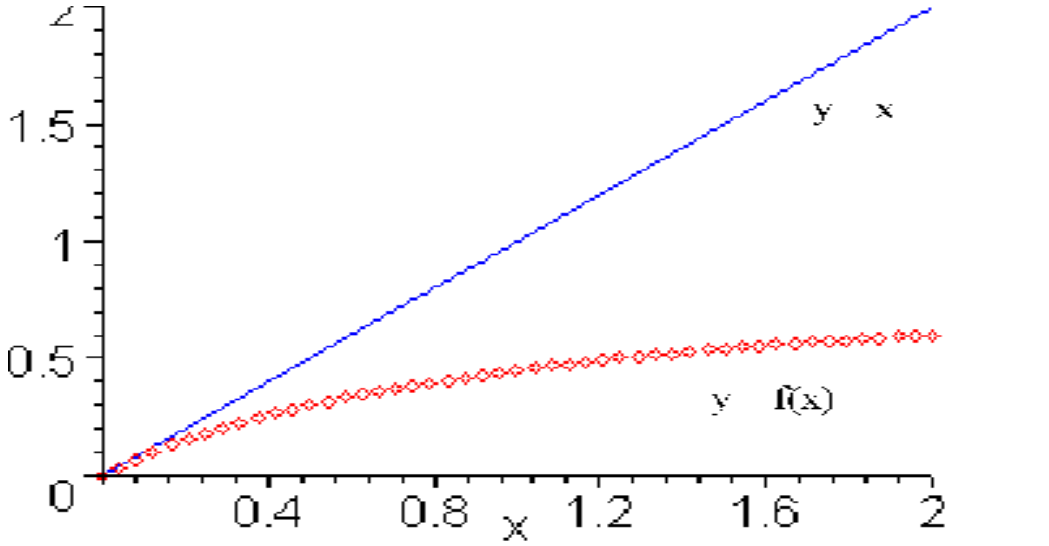
That is,

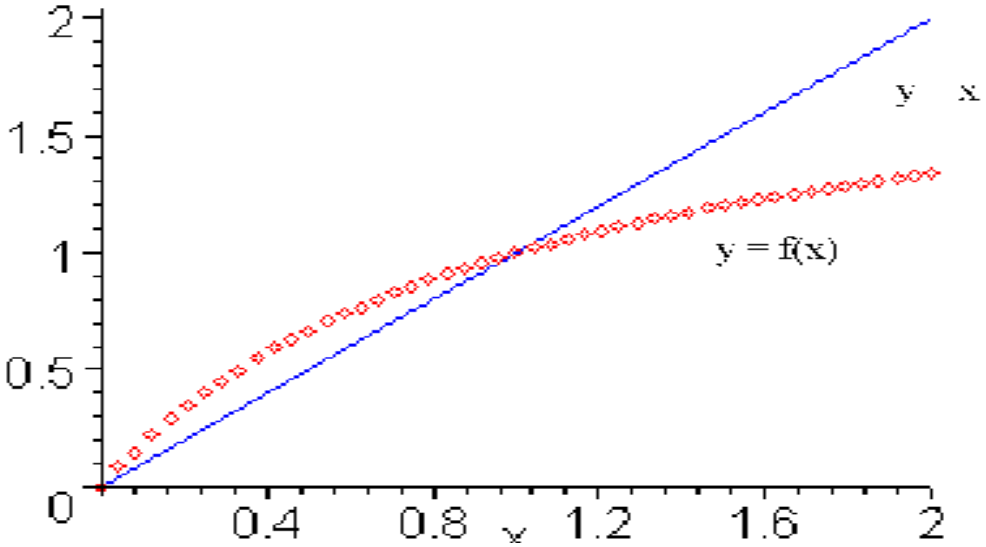
$T(n+1)=T(n)(g(T(n))+\gamma)$.

Demographic basic reproductive number is

$$R_d=g(0)/(1-\gamma)$$

Beverton-Holt Model



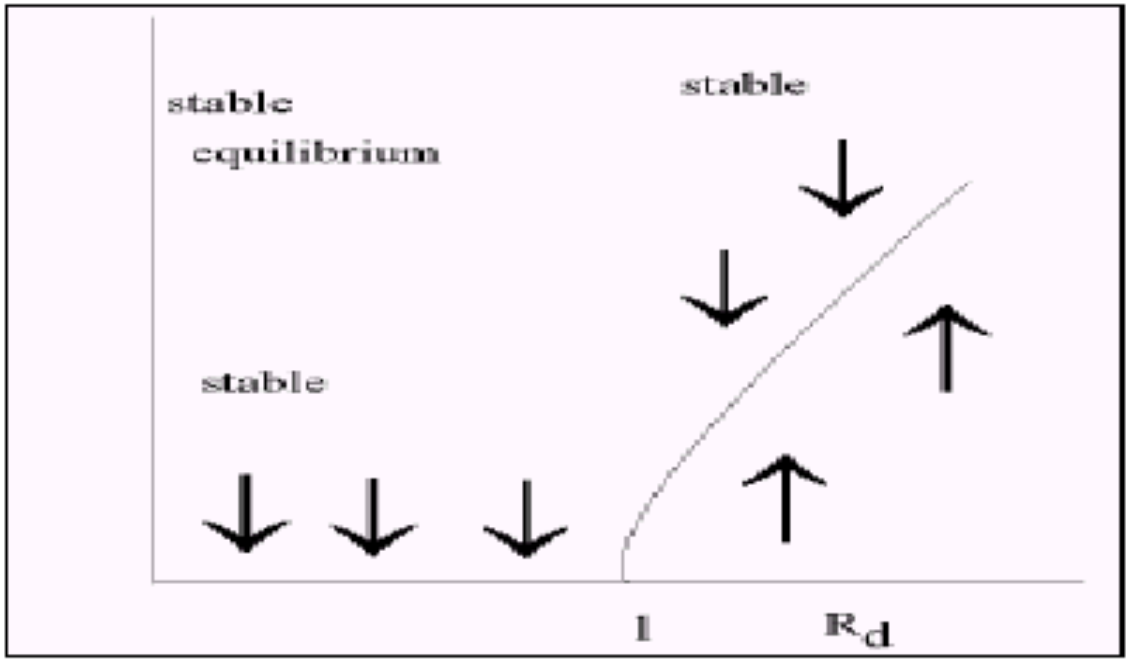


Example 1

Beverton-Holt Model:

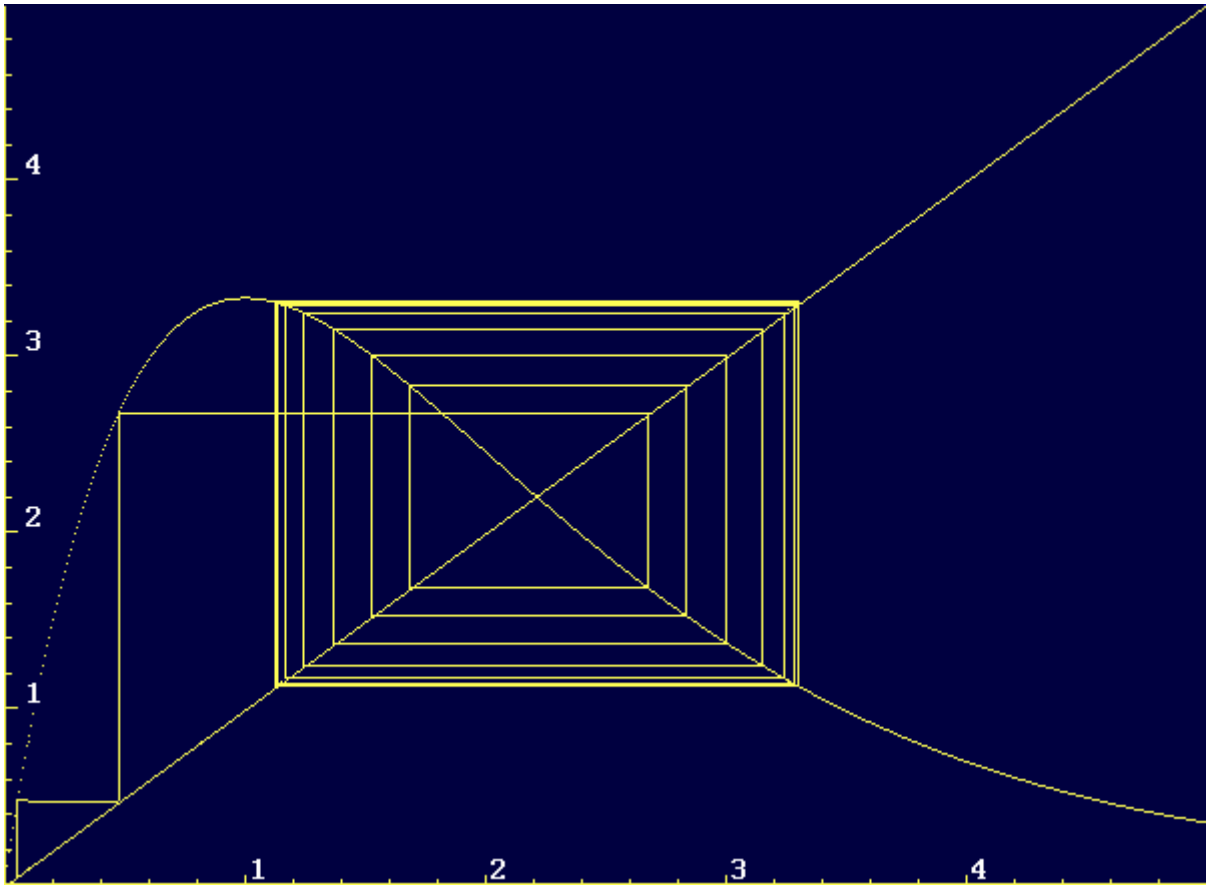
$$g(T(n)) = \frac{a}{1 + bT(n)}$$

$\Re_d > 1$ implies a globally stable positive fixed point at $T_\infty = \frac{1}{b}(\frac{a}{1-\gamma} - 1)$.



Fixed point dynamics supported

Ricker Model

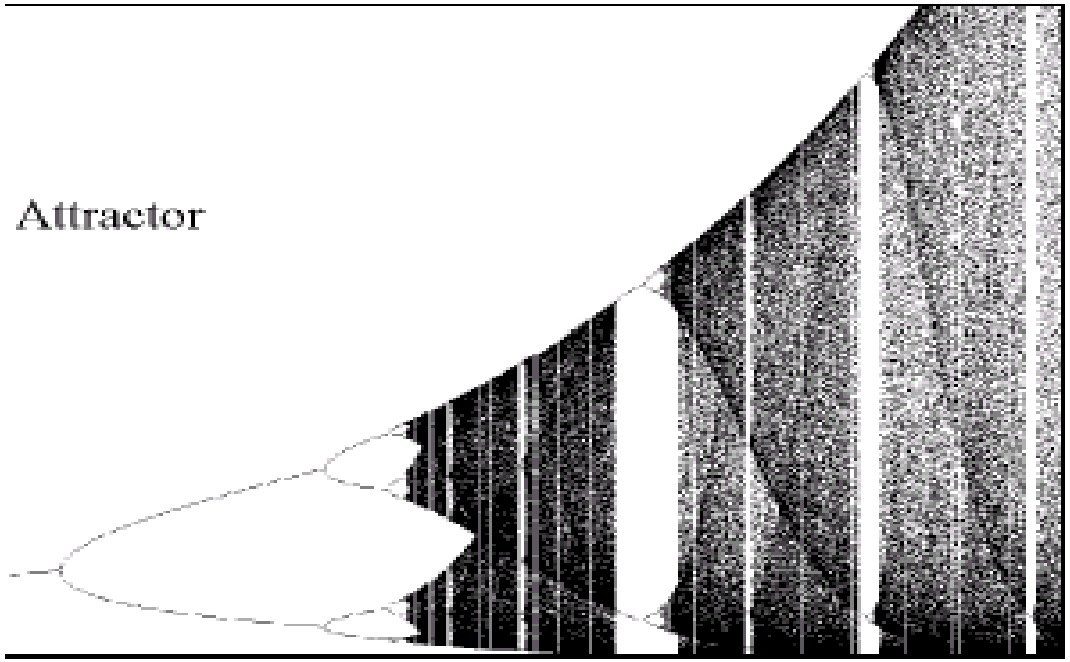


Example 2

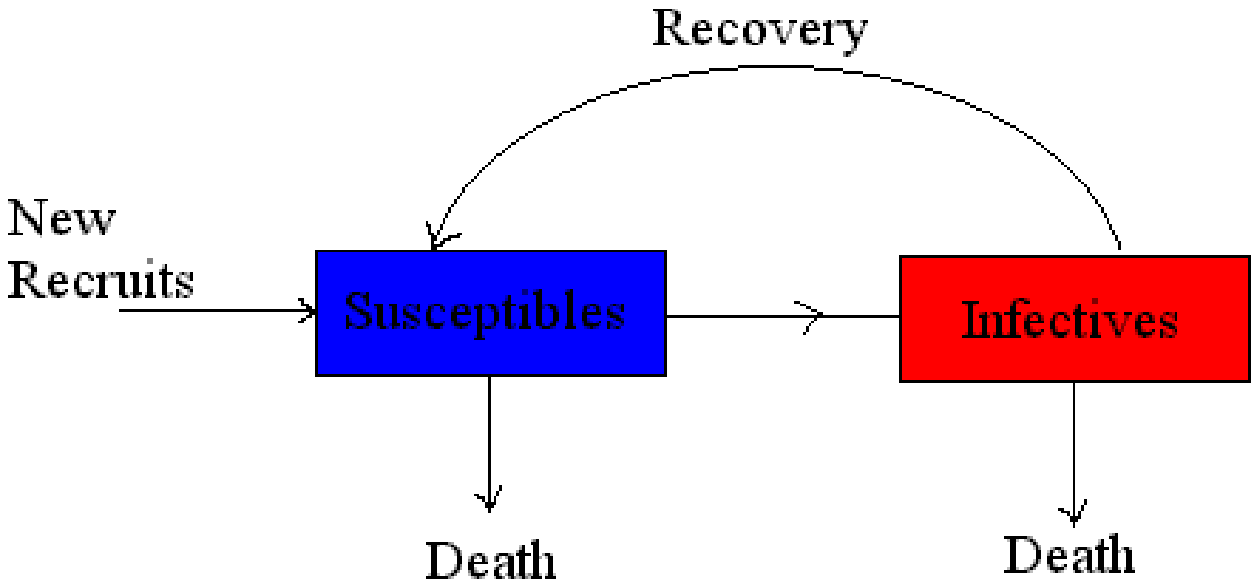
- Ricker's Model:

$$g(T(n)) = \exp(p - T(n))$$

Period-Doubling Bifurcation



S-I-S Model


Recovery
New
Recruits
Susceptibles
Infectives
Death
Death

Epidemic Model

At generation n,

- S(n): population of susceptible;
- I(n): population of infected;
- $T(n) \equiv S(n)+I(n)$: total population size;
- $T_\infty \equiv \lim_{n\to\infty} T(n)$: the demographic steady state;
- $f: [0, \infty) \to [0, \infty)$: denotes the recruitment function;
- γ: survival rate;
- $1-\sigma$: recovery rate.
- σ probability of surviving from the disease.

Infection Process

Susceptible individuals become infected with probability 1-G (remain susceptible with probability G).

$g(y) \equiv G(y\alpha(y))$ where the transmission function $\alpha \equiv \alpha(y)$ models the impact of prevalence ($y \equiv I/T$) on G.

In general, $G:[0, \infty) \rightarrow [0,1]$ is a monotone concave probability function with $G(0)=1$; $G'(x)<0$ and $G''(x) \geq 0$ for all x in $[0, \infty)$.

$\alpha :[0,1] \rightarrow [0, \infty)$ is a smooth function.

Model Assumptions

- When σ =1, the disease is fatal (the nonlinear population dynamics are fixed).
- The model assumes (implicitly) a sequential process. At the end of each generation, a fraction (1-γ) of each class is removed (death); susceptible then become infected with probability (1-G) and, independently, surviving infectives recover with probability (1-σ).

S-I-S Epidemic Model

$$S(n+1) = f(T(n))+\gamma\, g(y(n))S(n)+\gamma(1-\sigma)I(n),$$
$$I(n+1) = \gamma\,(1-g(y(n)))S(n)+\gamma\sigma I(n).$$

Questions

- What is relationship between model parameters and disease persistence or extinction?
- Can the infective population persist on a cyclic (non-equilibrium) attractor?
- What is the relationship between the population and epidemic attractors?

Asymptotically Bounded Growth

Assume that the total population has reached the positive steady state T_∞ and,

set $T(n) \equiv T_\infty$, $x(n)=S(n)/T_\infty$ and $y(n)=I(n)/T_\infty$ in System (2).

The resulting one-dimensional autonomous ``limiting system'' for y(n), is therefore given by

$$y(n+1) = \gamma\,(1-g(y(n)))(1-y(n))+\gamma\,\sigma\,y(n). \quad (3)$$

Limiting System

$$\left.\begin{array}{rcll} y(t+1) & = & G(y(t)), & y(0)=y\in R_+^n \\ z(t+1) & = & H(y(t),z(t)), & z(0)=z\in R_+^m \end{array}\right\}(*)$$

where $G: R_+^n \to R_+^n$ and $H: R_+^n \times R_+^m \to R_+^m$ are continuous functions.

Assumptions in System $(*)$:

1. Equation $y(t+1)=G(y(t))$ admits a globally attracting fixed point Y_∞ in R_+^n, and

2. $z(t+1)=H(Y_\infty, z(t))$ admits a globally asymptotically stable fixed point Z_∞ in R_+^m.

Theorem (BCY, 2002). *(Y_∞, Z_∞) in $R_+^n \times R_+^m$ is a globally attracting fixed point of System $(*)$.*

Global Stability

Theorem (Cull): *Suppose a continuous reproduction function* $f : [0,\infty) \to [0,\infty)$ *satisfies all the following properties:*

1. $$f(0) = 0.$$

2. *f has a unique positive fixed point, x_∞, satisfying*

   $$f(x) > x \text{ if } x < x_\infty,$$

   and

   $$f(x) < x \text{ if } x > x_\infty.$$

3. *If f has a maximum value at $m \in (0, x_\infty)$, then f is monotonically decreasing in $(m, f(m))$.*

Then x_∞ is a globally stable fixed point in the open interval $(0, f(m))$ if and only if $(0, f(m))$ contains no 2-cycles.

Basic Reproductive Number

The basic reproductive number, R_0, determines the asymptotic behavior of System (3).

$$R_0 = \begin{cases} \gamma\sigma & \text{if } \alpha(0)=1, \\ \dfrac{-\gamma\alpha(0)G'(0)}{1-\gamma\sigma} & \text{if } \alpha(0)\neq 1 \end{cases}$$

gives the average number of secondary infections generated by a small pioneer population of infected (assumed infectious) individuals over their life-time, whenever the disease is not fatal.

Global Stability (1)

Let $\alpha \equiv \alpha_0$ be a positive constant.

a) If $R_0<1$, then the solutions (x(n),y(n)) of System (3) approach the disease free equilibrium, (1,0), as $n \to \infty$.

b) If $R_0>1$,then the solutions (x(n),y(n)) of System (3) approach a unique positive endemic equilibrium, $(1-\hat{y}, \hat{y})$ in $(0, \infty) \times (0, \infty)$, as $n \to \infty$

Global Stability (2)

Let $\alpha(y) + y\,\alpha'(y) > 0$ and $2\alpha'(y) + y\,\alpha''(y) \leq 0$

a) If $R_0 < 1$, then the solutions $(x(n), y(n))$ of System (3) approach the disease free equilibrium, $(1,0)$, as $n \to \infty$.

b) If $R_0 > 1$, then the solutions $(x(n), y(n))$ of System (3) approach a unique positive endemic equilibrium, $(1-\hat{y}, \hat{y})$ in $(0, \infty) \times (0, \infty)$, as $n \to \infty$

Geometric Growth(1)

$x(n)=S(n)/T(n)$ and $y(n)=I(n)/T(n)$ reduces System (2) with $f(T(n))=\mu T(n)$:

$$x(n+1)=\mu/(\mu+\gamma)+ \gamma/(\mu+\gamma)\, x(n)g(y(n))+ \gamma/(\mu+\gamma)(1-\sigma)y(n),$$

$$y(n+1)= \gamma/(\mu+\gamma)\, x(n)(1-g(y(n)))+ \gamma/(\mu+\gamma)\, \sigma\, y(n).$$

$$x(n)+y(n)=1$$

Geometric Growth(2)

x(n)=1-y(n) reduces the System to a one-dimensional system of y(n):

y(n+1)= γ/(μ+γ) (1-y(n))(1-g(y(n)))+ γ/(μ+γ) σ y(n).

$$R_0 = \begin{cases} \dfrac{\gamma\sigma}{(1-R_d)\gamma + R_d} & \text{if } \alpha(0)=1, \\ \dfrac{-\gamma\alpha(0)G'(0)}{(1-\gamma)(R_d-1)+1-\gamma\sigma} & \text{if } \alpha(0)\neq 1 \end{cases}$$

Geometric Growth

$\Re_0$ is easily derived from the linearization of near $(x_\infty, y_\infty) \equiv (1,0)$, that is, from

$$y(n+1) \approx \frac{\gamma}{\mu + \gamma}(-a(0)G'(0) + \sigma)y(n).$$

If $\Re_d = 1$ (no demographic impact) then $\Re_0$ reduces to $\Re_0 = \gamma\sigma$ or $\Re_0 = \frac{-\gamma a(0)G'(0)}{1-\gamma\sigma}$ where $\frac{1}{1-\gamma\sigma}$ denotes the average death-adjusted length of the infectious period in generations; γ is the proportion of surviving susceptibles who can be invaded by the disease; and, $-a(0)G'(0)$ is the maximum rate of infection per infective. If $\Re_d \neq 1$ then demography impacts disease dynamics, that is $\Re_0$. In fact, $\frac{1}{(1-\gamma)(\Re_d - 1)+1-\gamma\sigma}$ gives the demographic death-adjusted infectious period measured in generations. Hence, $\Re_0$ decreases with population growth ($\Re_d > 1$) and increases with population decay ($0 < \Re_d < 1$) as all new recruits are assumed to be susceptibles.

Global Stability (1)

Let $\alpha(y) + y\,\alpha'(y) > 0$ and $2\alpha'(y) + y\,\alpha''(y) \leq 0$. Then,

(a) If $R_d<1$, the total population, $T \equiv S+I$, decreases to zero at a geometric rate; $R_d>1$ implies that the total population increases at a geometric rate; $R_d = 1$ implies that the total population remains fixed at its initial value.

(b) If $R_d > 1$ and $R_0<1$, then the proportion I/T of invectives in the total population tends to 0 as $n \to \infty$, while the proportion S/T of susceptible in the total population tends to 1 as $n \to \infty$.Hence, $(S/T, I/T)$ tends to the disease-free equilibrium (1,0), where S is increasing at the same geometric rate as T.

Global Stability (2)

(c) If $R_d > 1$ and $R_0 > 1$, then the proportion I/T of invectives in the total population tends to a positive number $\bar{I}/T$ as n approaches infinity, and the proportion S/T of susceptible in the total population tends to a positive number $(1 - \bar{I}/T)$ as n approaches infinity. Hence, (S/T,I/T) tends to an endemic. I, S, and T are increasing at the same geometric rate.

(d) If $R_d < 1$ and $R_0 < 1$, then the proportion I/T of invectives in the total population tends to 0 as n approaches infinity, while the proportion S/T of susceptible in the total population tends to 1 as n approaches infinity. Hence, (S/T,I/T) tends to disease-free equilibrium. Hence S is increasing to zero at the same geometric rate as T.

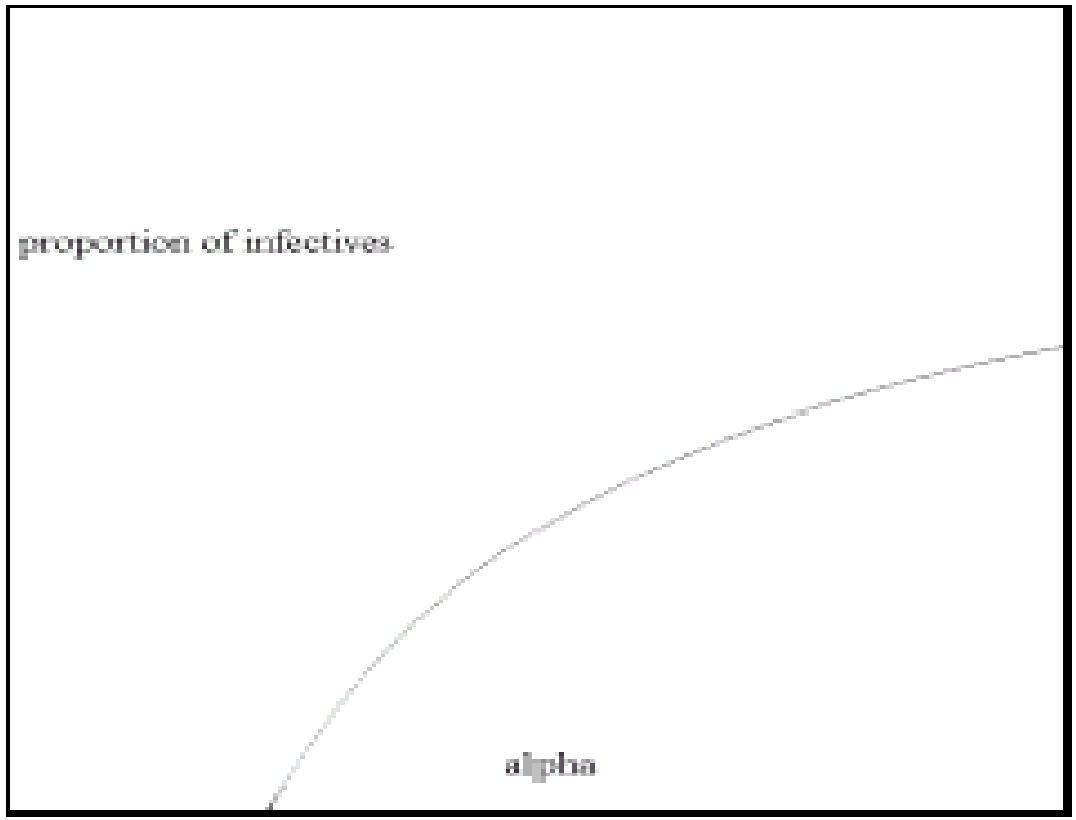
(c) If $R_d < 1$ and $R_0 > 1$, then the proportion I/T of invectives in the total population tends to a positive number $\bar{I}/T$ as n approaches infinity, and the proportion S/T of susceptible in the total population tends to a positive number $(1 - \bar{I}/T)$ as n approaches infinity. Hence, (S/T,I/T) tends to an endemic. I, S, and T are increasing at the same geometric rate.

Illustrative Example (1)

- $e^{-d}=\gamma$, $e^{-\alpha I(n)/T(n)} = G(\alpha I(n)/T(n))$ and $e^{-\beta}=\sigma$.
- $f(T(n))=\mu T(n)$ implies that $T(n+1)= (e^{-d}+\mu)T(n)$ and

  $R_d=\mu/(1- e^{-d})$
- $R_0=\alpha^2/(1+\mu e^{-d}- e^{-\beta})$
- $\beta=0.1$, $d=\ln 2$ and $\mu=0.1$ are fixed and the transmission coefficient α is varied.

Illustrative Example (2)



Questions

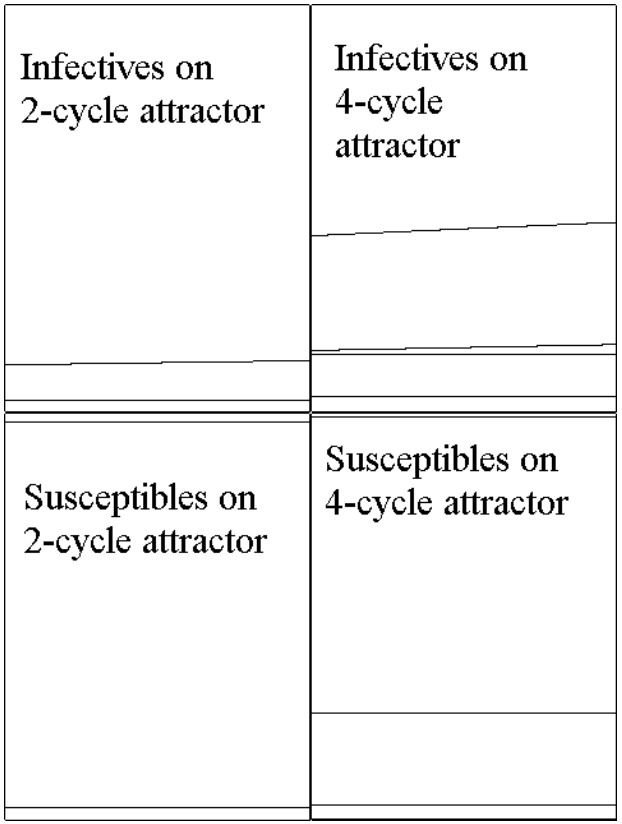
- Can complex demographic dynamics drive disease dynamics?

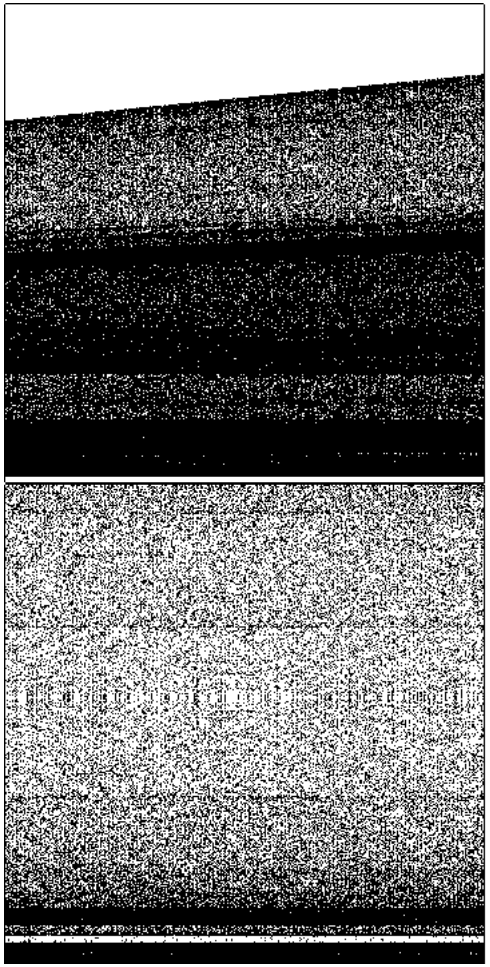
- Are simple discrete-time epidemic models bistable?

K-Cycle Attractors

- S- is on a hyperbolic attracting k-cycle whenever the T- is on a hyperbolic attracting k-cycle and $\alpha = 0$.
- I- is on a positive k-cycle whenever both the T- and S- are on different k-cycles.
- Assume that $\{\check{z}_0(0), \check{z}_1(0), \ldots, \check{z}_{k-1}(0)\}$ is a hyperbolic k-cycle for (SIS) when $\alpha = 0$. Then, for α close enough to 0, there is a curve of k-cycles for (SIS) given by $\{\check{z}_0(\alpha), \check{z}_1(\alpha), \ldots, \check{z}_{k-1}(\alpha)\}$. This k-cycle is an attractor for (SIS) if $\{\check{z}_0(0), \check{z}_1(0), \ldots, \check{z}_{k-1}(0)\}$ is a hyperbolic attracting k-cycle when $\alpha = 0$.

S-Dynamics Versus I-Dynamics





Infectives on a chaotic attractor

Susceptibles on a chaotic attractor

S-E-I-S MODEL

$$\left.\begin{aligned} S(n+1) &= f(T(n)) + \gamma G(\varepsilon(n)) S(n) + \gamma(1-\delta) I(n), \\ E(n+1) &= \gamma(1 - G(\varepsilon(n))) S(n) + \gamma\sigma E(n), \\ I(n+1) &= \gamma(1-\sigma) E(n) + \gamma\delta I(n). \end{aligned}\right\}$$

S-I-S Epidemic Models With Delay

$$S(t+1)=f(T(t-k))+\gamma S(t)G(\alpha I(t)/T(t))+\gamma I(t)(1-\sigma),$$

$$I(t+1)=\gamma(1-G(\alpha I(t)/T(t)))S(t)+\gamma\,\sigma\, I(t)$$

Demographic equation becomes

$$T(t+1)=f(T(t-k))+\gamma T(t)$$

Geometric Growth

If new recruits were to arrive at a constant per-capita rate μ then

$$f(T(n-k)) = \mu T(n-k)$$

and the demographic equation becomes

$$T(n+1) = \mu T(n-k) + \gamma T(n).$$

This last equation has geometric solutions of the form

$$T(n) = T(0)\lambda^n$$

where $\lambda > \gamma$ is a solution of the characteristic equation

$$\lambda = \frac{\mu}{\lambda^k} + \gamma$$

and $T(0) \neq 0$. In fact, whenever λ^* is the unique largest real solution of the characteristic equation then $\lambda^* \geq |\lambda|$ (where λ is any real or complex solution of the characteristic equation).

In fact, $\gamma < \lambda^* < 1$ implies $\lambda^* > \sqrt[k]{\frac{\mu}{1-\gamma}}$ while $\gamma < 1 < \lambda^*$ implies $\lambda^* < \sqrt[k]{\frac{\mu}{1-\gamma}}$. Consequently, the demographic basic reproductive number is defined by

R_d

$$R_d = \sqrt[k]{\frac{\mu}{1-\gamma}}$$

R_d a dimensionless quantity, gives the average number of susceptible produced by a (typically small) pioneer population (T(0)) over its life-time. The k^{th}-root accounts for the fact that a T(0) descendant must survive k-generations before it joins the population of susceptible.

R_0

We assume that the T-population has been around long enough (prior to disease invasion) for $T(t) = T(0)(\lambda^*)^t$, where $\lambda^* > |\lambda|$. Rescaling reduces the System with new recruits under geometric growth ($T(t) = T(0)(\lambda^*)^t$) to the following one-dimensional autonomous ``system'' for y(t):

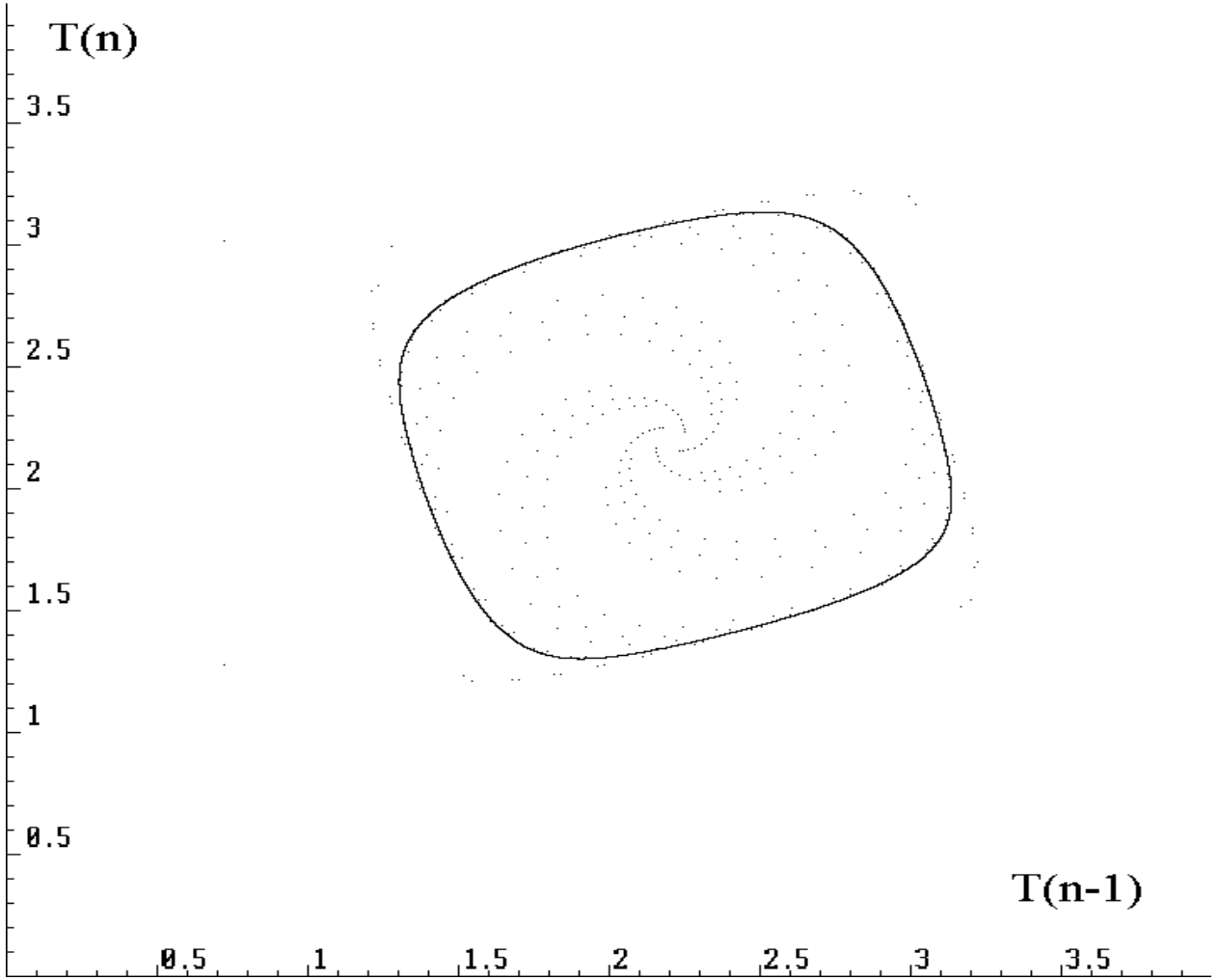
$$y(t+1)=(\gamma/\lambda^*)(1-y(t))(1-G(\alpha y(t)))+(\gamma\sigma/\lambda^*)\, y(t)$$

$$R_0 = \frac{-\alpha G`(0)(\gamma/\lambda^*)}{1-(\gamma\sigma/\lambda^*)}$$

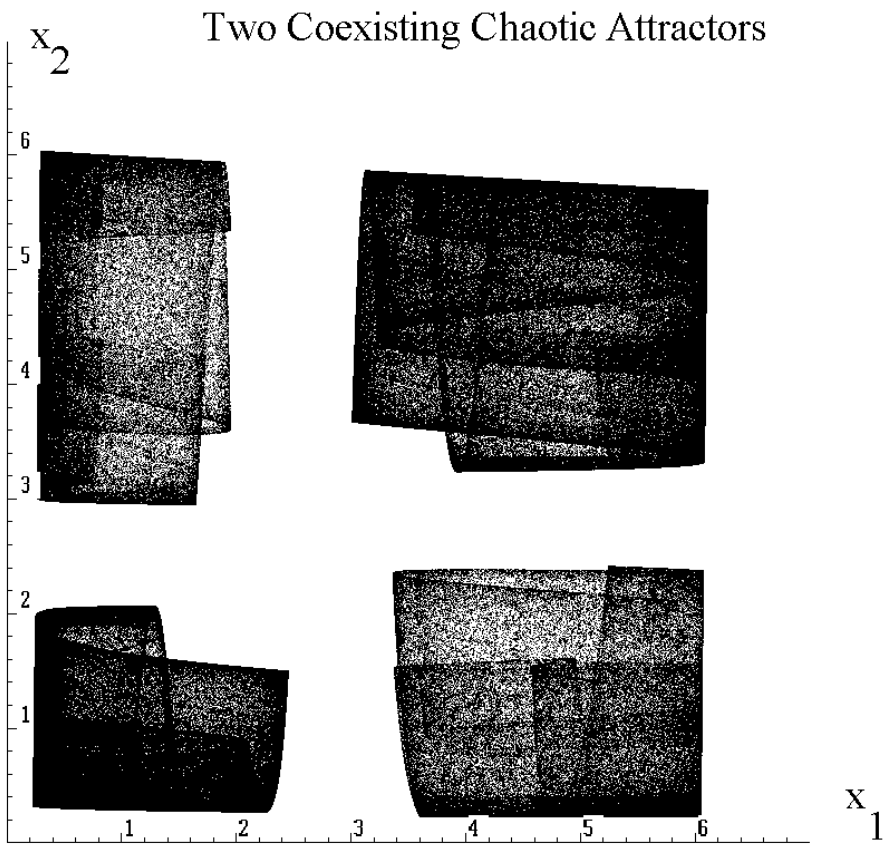
T-Dynamics (1)

If new births or new recruits are governed by
$f(T(n-k))=T(n-k)\exp(r-T(n-k))$
then the presence of delay increases the level of dynamic complexity in the T- dynamics. We can now keep the T- dynamics on a selected Hopf invariant closed curve or a strange (chaotic) attractor.
The equation
$T(n+1)=T(n-1)\exp(r-T(n-1))+\gamma T(n)$
has a positive equilibrium that is capable of undergoing a discrete-time Hopf bifurcation.

T-Dynamics (2)



T-Dynamics (3)

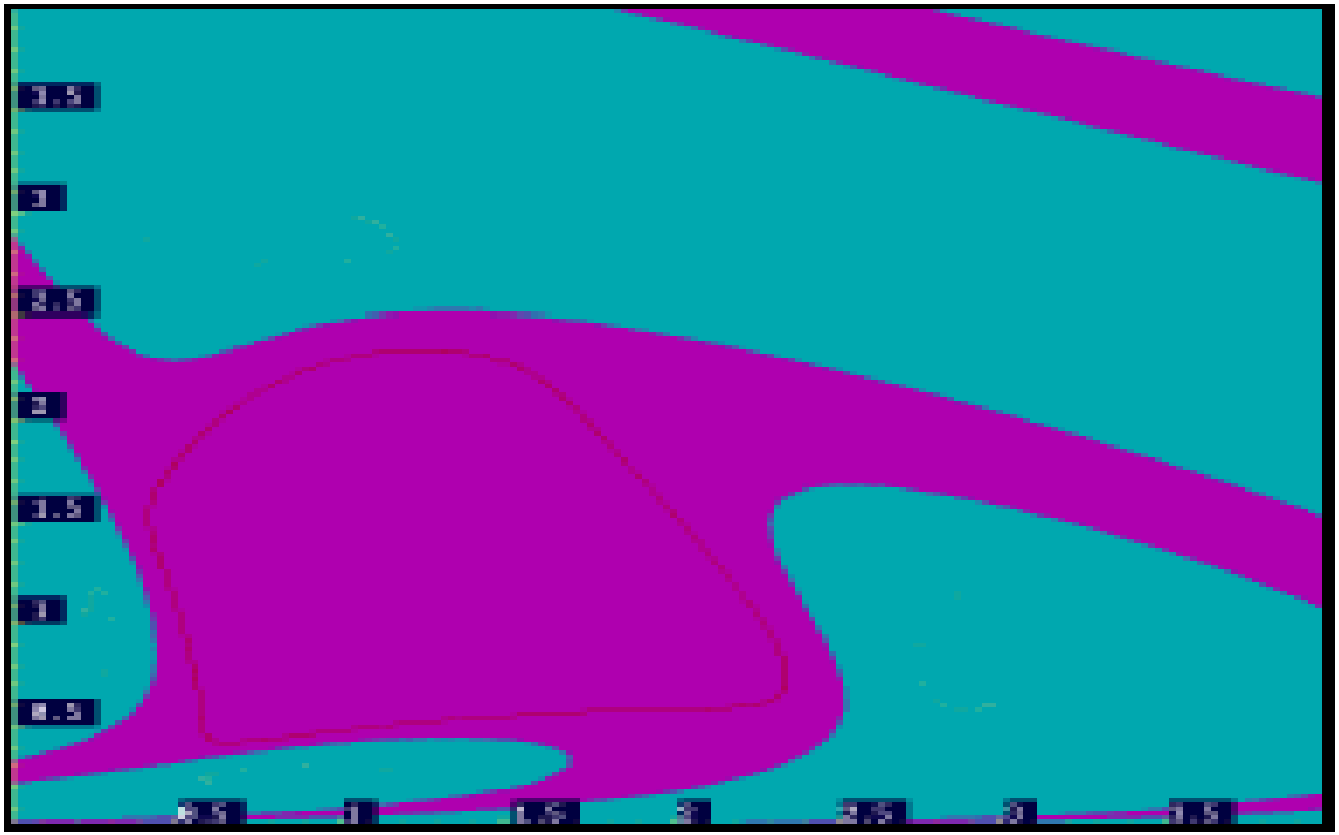


Bistability in T-Dynamics (1)

- $T(t+1)=f(T(t-k),T(t-(k-1)),\ldots,T(t)), + \gamma T(t)$

- The equation
- $T(t+1)=T(t)\exp\{r-cT(t-1)-T(t)\} + \gamma T(t)$
- Supports multiple attractors when c=0.9, r=2.3,
- k=1, and γ=0.01

Bistability in T-Dynamics (2)



Dynamical Systems Theory

- Equilibrium Dynamics, Oscillatory Dynamics, Stability Concepts, etc
- Attractors and repellors (Chaotic attractors)
- Basins of Attraction
- Bifurcation Theory (Hopf, Period-doubling and saddle-node bifurcations)
- Perturbation Theory (Structural Stability)

Conclusions

- Complex internally driven population dynamics can ``drive'' disease dynamics. Hence, disease is likely to have a dramatic impact on local life-history evolution even when it is non-fatal.
- Contact rate and dispersal play key roles on diseases survival and epidemic severity.
- Age-structure expands the class of attractors where epidemics can live.

Discussion

- Developing more realistic epidemic processes and adding the impact of disease induced mortality leads to a class of challenging nonlinear systems

- Role of dispersal, population dynamics, and disease on life history evolution

- Connections with real data.

SIS Epidemic Model With Disease-Induced Death

$$\left.\begin{aligned} S(t+1) &= f(N(t)) + \gamma_1\phi(\alpha\frac{I(t)}{N(t)})S(t) + \gamma_2(1-\sigma)I(t) \\ I(t+1) &= \gamma_1\left(1-\phi(\alpha\frac{I(t)}{N(t)})\right)S(t) + \gamma_2\sigma I(t) \end{aligned}\right\} \quad (3)$$

where $0 < \gamma_2 < \gamma_1 < 1, 0 < \sigma < 1$ and $N(t) > 0$.

The escape function $\phi : [0,\infty) \to [0,1]$ is a monotone convex probability function with $\phi(0) = 1$ and $\phi' \leq 0$.

Model Assumptions

- Disease increases mortality but does not affect fecundity;
- No acquired immunity;
- No latent period (or latent period is very short);
- Transmission is frequency dependent rather than density dependent.

Deterministic SIS Model

- Our model is a deterministic SIS epidemic model and has no “probability” of transmission. The assumption of deterministic dynamics is valid in a large population, where stochasticity is unimportant.
- This assumption places a constraint on the applicability of our model. For example, stochastic transmission (including a Poisson process) in a small population (close to extinction) would not be described by our model.

Disease Extinction or Persistence

Let $R_0 = \dfrac{-\gamma_1 \alpha \phi'(0)}{1-\gamma_2 \sigma}$.

No disease induced death : Castillo – Chavez and Yakubu [2001]

Theorem (Franke and Yakubu, 2008) :

Let $N(0) \geq I(0) > 0$.

1. If $R_0 < 1$, then $\lim_{t\to\infty} I(t) = 0$. That is, the disease goes extinct.

2. If $R_0 > 1$ and the total population is uniformly persistent , then ther e exists $\eta > 0$ such that $\underline{\lim}_{t\to\infty} I(t) \geq \eta > 0$. That is, the disease is uniformly persistent .

R_0

- Without disease-induced mortality, it is known that $R_0 > 1$ implies disease persistence.
- With disease-induced mortality, independent of initial population size of healthy individuals, a tiny number of infectious individuals can drive the total population to extinction.

Auxiliary Functions

1. $D_i(N) = f(N) + \gamma_i N$

The total population of new births and survivors;

2. $\mathrm{F_N}(I) = \gamma_1(1-\phi(\alpha\frac{I}{N}))(N-I) + \gamma_2\sigma I$

Infective population in the next generation;

3. $\mathrm{G_N}(I) = f(N) + \gamma_1(N-I) + \gamma_2 I$

Total population in the next generation;

4. $\mathrm{H(N, I)} = (\mathrm{G_N}(I), \mathrm{F_N}(I))$

Vector of the total and infective populations.

Disease-Free State

If $I(t) = 0$, then the demographic equation

$$N(t+1) = f(N(t)) + \gamma_1 S(t) + \gamma_2 I(t)$$

becomes

$$S(t+1) = f(S(t)) + \gamma_1 S(t).$$

This reduced equation describes the disease-free state dynamics.

Demographic Basic Reproduction Number

$$R_{D_i} = \frac{f'(0)}{1-\gamma_i} \text{ whenever } f(0) = 0.$$

1. Let $f(0) = 0$. If $R_{D_1} > 1$, then the disease-free susceptible population is persistent.
2. Let $f(0) = 0$. If $R_{D_1} < 1$, then $\{(0,0)\}$ is locally asymptotically stable. That is, both the susceptible and infected populations go extinct at low population sizes.
3. R_{D_1} is the disease-free state demographic basic reproduction number.
4. If either $f(0) > 0$ or $f(0) = 0$ and $R_{D_2} > 1$ then the total population is uniformly persistent.

Dramatic Population Extinction

Theorem : Let $R_0 > 1$, $f(0) = 0$ and $f(N) \leq f'(0)N$

for all $N > 0$. Then there is a function

$\zeta = \zeta(\gamma_1, \gamma_2, \phi, \alpha, \sigma, F_1) > 1$ such that if $1 < R_{D_1} < \zeta$

then the total population goes extinct under H iterations.

Illustrative Example

Let $f(N) = \dfrac{aN}{1+bN}$ and $\phi(N) = e^{-\frac{\alpha I}{N}}$

where

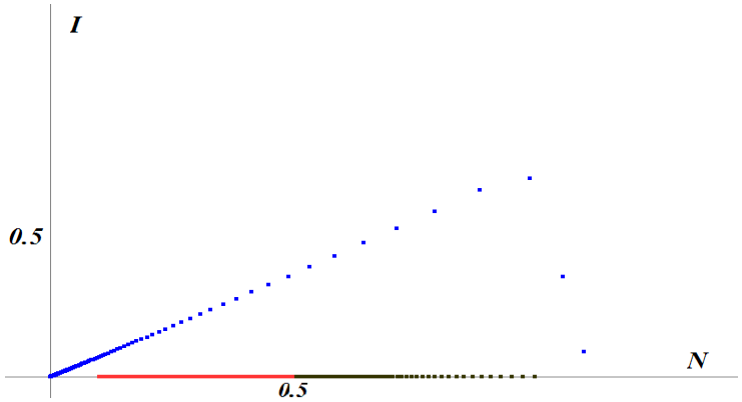
$0.1 < a < 0.2$, $b = 1$, $\alpha = 5$, $\gamma_1 = 0.9$,

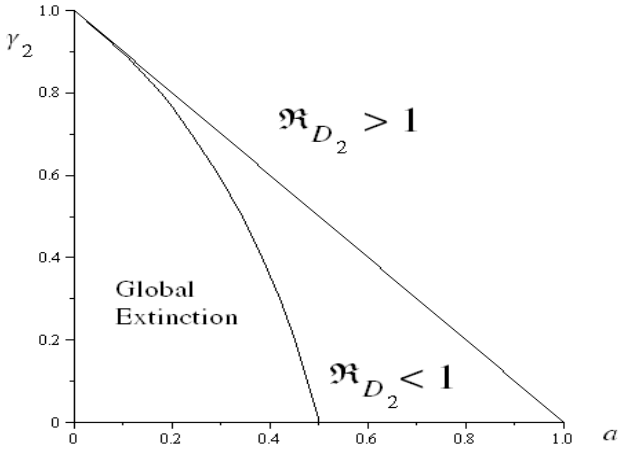
$\gamma_2 = 0.8$ and $\sigma = 0.9$.

$R_{D_1} = \dfrac{a}{1-\gamma_1} > \dfrac{0.1}{1-0.9} = 1$ implies the persistenc e of the susceptibl e population in the absence of the disease.

$R_{D_2} = \dfrac{a}{1-\gamma_2} < \dfrac{0.2}{1-0.8} = 1.$

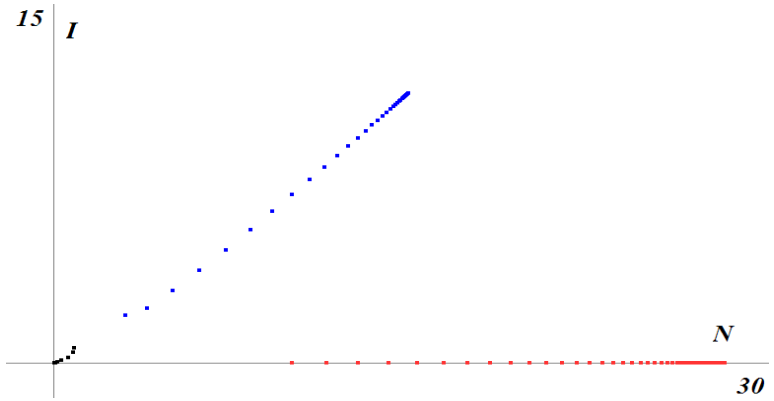
As predicted by the theorem, $0.1 < a < 0.177$ gives the extinction of the total population .



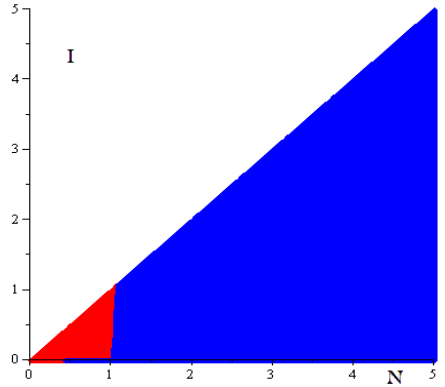


Multiple Attractors

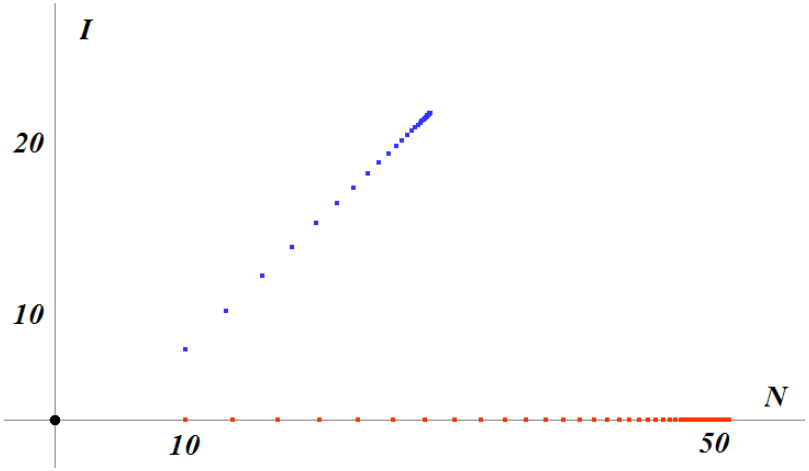
Theorem : Let $\overline{\lim}_{N\to\infty} \frac{f(N)+\gamma_1 N}{N} < 1$ and $R_{D_2} > 1$. Then H has multiple fixed points when G_N "decreases" at low population sizes while it "increases" at high population values.

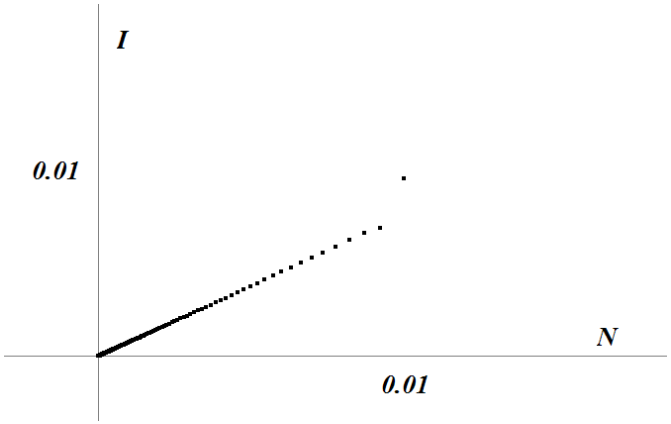


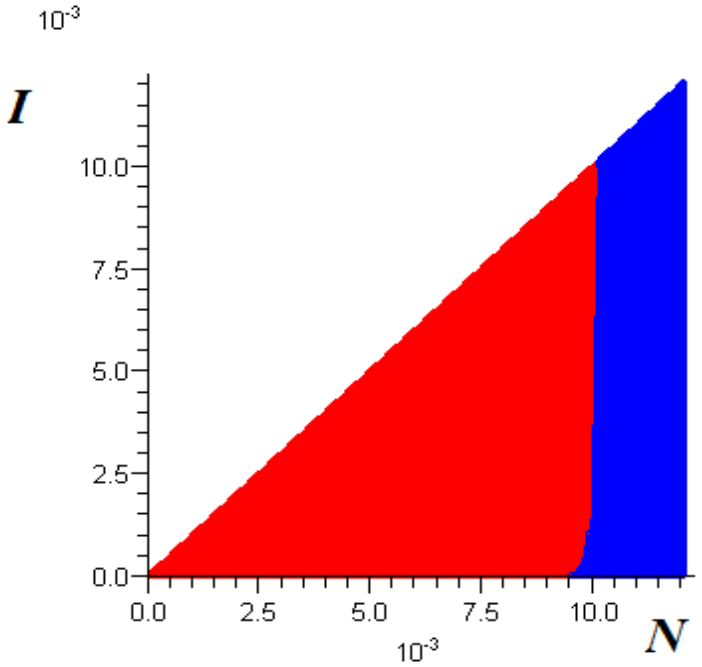
Corollary : Let $\overline{\lim}_{N\to\infty} \frac{f(N)+\gamma_1 N}{N} < 1.$ If $R_{D_1} > 1$ and there is $0 < N_0$ with $G_{N_0}(I_1 N_0) > N_0$, then the origin is not a global attractor and H has at least two positive fixed points.



Origin an Attractor







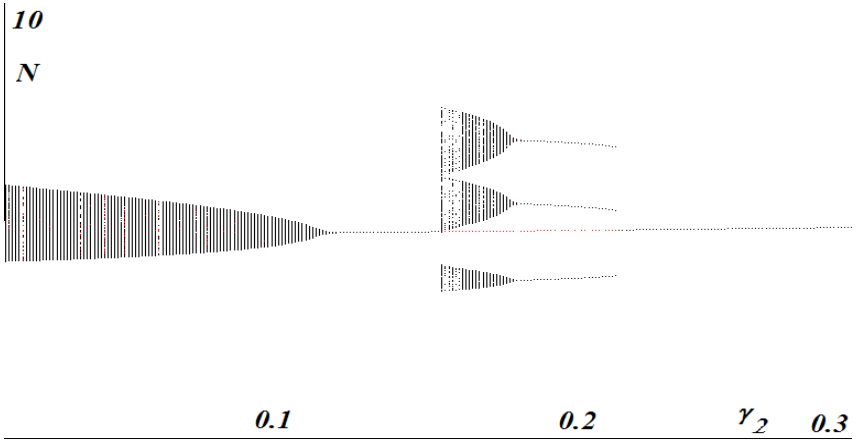
Complex Disease Dynamics

Let $f(N) = N\exp(r - N)$ and $\varphi(\frac{\alpha I}{N}) = e^{-\frac{\alpha I}{N}}$

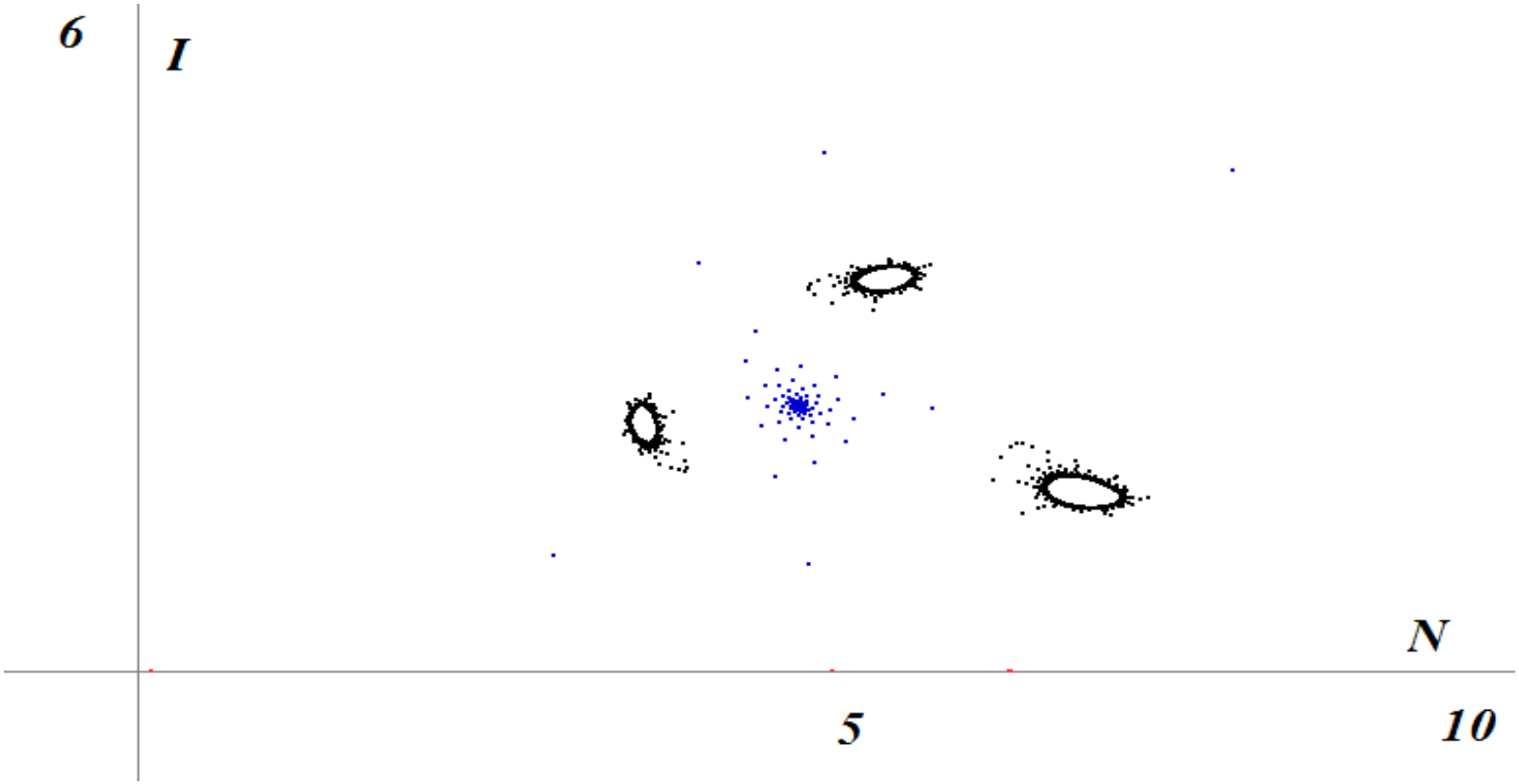
where

$\alpha = 5$, $\gamma_1 = 0.9$, $\gamma_2 \in (0, 0.9)$, $r = 4$ and $\sigma = 0.9$.

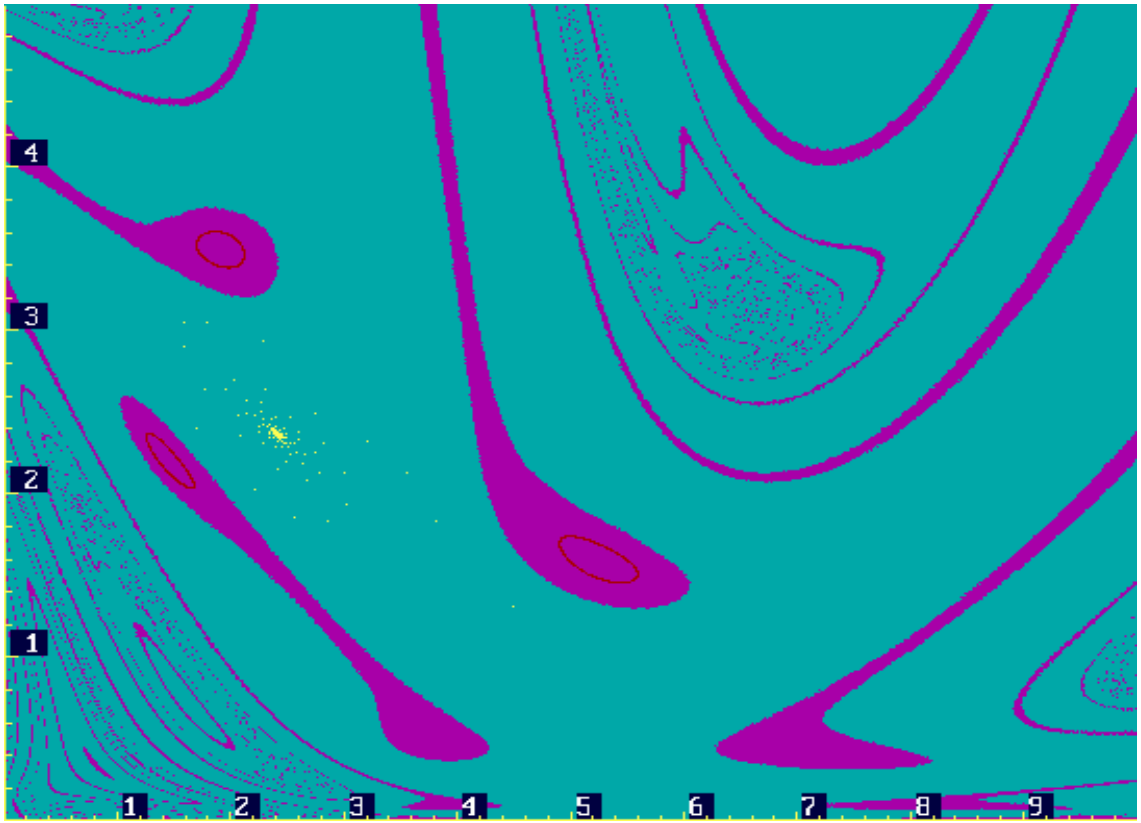
In the absence of the disease, the susceptibl e population is on a globally attracting positive fixed point at $S_\infty = 6.303$.



Overcompensatory Dynamics



Fractal Basin Boundaries



Geometric Growth

Let $f(N) = \mu N$. In the absence of the disease, the suceptible (disease - free state) equation becomes

$$S(t+1) = \mu S(t) + \gamma_1 S(t) = (\mu + \gamma_1) S(t).$$

Hence,

$$S(t) = (\mu + \gamma_1)^t S(0) \quad \text{and} \quad R_{D_1} = \frac{\mu}{1 - \gamma_1}.$$

SIS Model With Geometric Growth

Let $i = \frac{I}{N}$ and $s = \frac{S}{N}$.

Then $i(t) + s(t) = 1$ and our SIS model becomes

$$i(t+1) = \frac{F_1(i(t))}{\mu + \gamma_1 + (\gamma_2 - \gamma_1)i(t)} \qquad (5)$$

R_0

Under geometric growth,

$$R_0 = \frac{-\gamma_1 \alpha \varphi'(0)}{(1-\gamma_1)(R_{D_1} - 1) + 1 - \gamma_2 \sigma}.$$

Theorem : If $R_0 \leq 1$, then $\lim_{t \to \infty} i(t) = 0$.
That is, the proportion of the infected eventually decreases to zero.

If $R_0 > 1$, then the proportion of the infected population is uniformly persistent .

Envelopes on Compact Intervals [Cull, 1986]

Let $F : [0,1] \rightarrow [0,1]$ have a unique critical point, i_c, and a unique positive fixed point, i_∞, where $0 < i_c < i_\infty < 1$. Also, let $\{0\}$ be an unstable fixed point of F.

A function $E : [0,1] \rightarrow [0,1]$ envelopes the function F if and only if

$E(i) \geq F(i)$ on $[0, i_\infty]$ and

$E(i) \leq F(i)$ on $[i_\infty, 1]$.

Globally Stable Positive Fixed Point

Theorem (Cull [1986]) : If E envelopes F on [0,1] and $E(E(i)) > i$ for all i in $[i_c, i_\infty)$, then i_∞ is a globally asymptotic ally stable positive fixed point of F on (0,1].

Theorem [F - Y, 2008] : If $R_0 > 1$, our SIS epidemic model with geomteric growth has a unique positive globally asymptotic ally stable equlibrium .

Conclusion

- Our model framework allows the population dynamics and disease transmission to be fairly general.
- We highlighted the role of disease-induced mortality, and the complexity of the interaction between infectives and susceptibles in discrete-time models.
- Disease-induced death can force the extinction of a population with $R_0 > 1$, where the population persists without disease-induced death.
- Disease-induced death can generate multiple attractors with complicated basin structures.
- In epidemic models with disease-induced death, the disease-free dynamics do not drive the disease dynamics.

Thank You!